FUZZY LOGIC AND NON-DISTRIBUTIVE TRUTH VALUATIONS

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ABSTRACT

In 1936, Birkoff and von Neumann were able to show that a non-distributive, orthocomplemented, modular lattice was equivalent to the traditional mathematical representation of quantum mechanics. In such a lattice of propositions, it is demonstrable that there can be no coherent truth valuation set of cardinality greater than two inasmuch as a finite probability measure is not homomorphic with a non-distributive valuation on the real interval [0,1]. In the present paper, it is empirically demonstrated that the logic of natural languages is non-distributive. Thus, the lattice theoretic representation reduces from the traditional Boolean one to that proposed by Birkoff and von Neumann for quantum mechanics, commonly known as quantum logic. This result implies that fuzzy logics, probability logics, and multi-valued logics are inappropriate representations of either natural linguistic or quantum mechanical propositions. At best, such representations are valid only under limited conditions in which the lattice is locally Boolean (known as the isles of Boole). A criteria is presented for the determination of these conditions from empirical data.

INTRODUCTION

The application of formal systems to empirical situations (modeling) is a difficult task. It is easy to confuse the properties of the formal system with those of the empirical system. In order to avoid such errors, a consistent modeling methodology must be pursued and we have outlined such a methodology in another paper [14]. A part of this methodology

entails verification of the consistency of the model in the light of a given set of observations. We pursue that task in this paper.

There exists within the literature considerable discussion of the various applications of fuzzy logic. The extensive bibliography of Kandel and Byatt [4] lists 566 papers on fuzzy logic. A number of these are concerned with the applications of fuzzy logic to cognitive processes such as memory [5], [8], [9], [10], [11], [16], psycholinguistics [6], and language [12], [15], [20]. The sucessful application of fuzzy logic to any system rests upon the premise that the abstract algebraic structure of system observables is indeed isomorphic to the abstract algebraic structure of fuzzy logic. In most applications given in the literature, this isomorphism is assumed in a general way via a heuristic argument.

It will be demonstrated that linguistic phenomena do not obey the distributive or commutative laws (see Watanabe [19] who anticipated this as early as 1959) and thus, since the usual fuzzy logic definitions of intersection and union entail both the distributive and commutative laws [3], the isomorphism fails. It is argued (though not proven) that this result applies to all cognitive processes. Finally, it is proven that, if the fuzzy logic definitions of union and intersection are changed so that the distributive and commutative laws are no longer part of the formal system, the formal logic is then incoherent for the usual truth valuation set [0,1].

FUZZY LOGIC: EMPIRICAL OR FORMAL

The very nature of mathematics and logic is deductive. As such, a any algebra or logic has a certain hierarchical structure. Starting from the axioms of the system, a large number of propositions can be derived, but only because a strong caveat holds: the axioms and primitive propositions must be consistent. Is fuzzy logic internally consistent?

In order to investigate this question, we take the position that

there are two quite distinct fuzzy logics: one which is conserned only with the system as an abstract mathematical structure (which we shall refer to as formal fuzzy logic) and one which is conserned with the applicable results of fuzzy logic (which we refer to as empirical fuzzy logic). In a recent paper [18], Watanabe gave three major shortcomings of fuzzy set theory as proposed by Zadeh [23], which we take here as an example of formal fuzzy logic. 1) The value of the membership function cannot be rationally determined. 2) The implication relation is strictly dependent upon this determination for the two membership functions involved. 3) The minimum-maximum rules for conjunction and disjunction are completely arbitrary.

In addition to a generally unsolvable difficulty which we shall detail shortly, the first complaint also applies to empirical fuzzy logic: not only is it not possible to determine the membership rationally, it cannot be determined empirically either. The question of quantification in empirical science is easily understood from the point of view of operational positivism. Abstractly, quantification entails 1) a model, 2) an interpretation, 3) a standard measurement procedure, and 4) a standard. The model used in any quantification process is a set of suppositions about the relationship between the quantity to be measured and the standard procedure established to accomplish the measurement. The interpretation of the model establishes a one-to-one correspondence between the variables of the model and the observables. The standard is a physical system which, when me sured according to the standard procedure, results in a specific quantity which is consistent with all prior measurements. Quantification, therefore, is other than arbitrary only inasmuch as the empirical system constrains the choice of a model. Applied to empirical fuzzy logic, this representation of quantification implies that a unique membership function corresponds to a unique choice

of a general model for all "fuzzy" variables. We know of no develop-
ments along these lines and, at present, consider the pursuit of such a
unique model of dubious worth.

If we do not pursue a unique membership function, empirical fuzzy
logic is relegated to the realm of a mathematical curiosity. To define
a membership function for each class of fuzzy variable is also disas-
trous to the intended power of the theory. For example, it is useful to
compare the relative truth of two statements such as "John is tall" and
"Dick is tall" or perhaps the two statements "John is a liar" and "Bill
is a liar" via the truth values of either the conjunctions or disjunc-
tions. The truth values of "John is tall and Dick is tall" and of "John
is tall or Dick is tall" can be determined empirically if the fuzzy
variable "tallness" is quantified. Similarly, the truth value of "John
is a liar and Bill is a liar" can be determined empirically if the fuzzy
variable "liar" is quantified. In each case we will be led to an opera-
tional (empirical) definition of the membership function.

Although these cases are of considerable interest, the logic is too
limited when we cannot also make predictions about the truth values of the
conjunction "John is a liar and Dick is tall." In other words, since the
membership function must be determined empirically and since the opera-
tional procedure for quantification leads to semantically dependent defi-
nitions of the membership functions, the notion of an empirical fuzzy
logic fails.

We agree, in general, with the opinions expressed by Watanabe that
a new theory, if interpreted at all, should bring about "new empirically
verifiable conclusions that the old theory could not yield." We do not
agree, however, that fuzzy logic should be reduced to Boolean logic.
Such an attempt affords us nothing; the fuzziness was introduced with the
hope that inexact concepts might be treated more rigorously (and realis-
tically), but that same fuzziness yields nothing new when fuzzy logic is
reduced or generalized to the usual Boolean logic. We have thus come

full circle.

THE AXIOMATIC STRUCTURE OF EMPIRICAL FUZZY LOGIC

The usual definitions of Zadehian fuzzy set theory, and in particular the simple definition given for the implication "→", dictate a lattice structure for the set F of all fuzzy sets A which satisfies 1) the idempotent law, 2) the associative law, 3) the commutative law, 4) the absoptive law, and 5) the distributive law.

Kandel and Byatt define a fuzzy algebra to be a system

$$Z = (\underline{Z}, +, *, \rightarrow)$$

where $\underline{Z}$ has at least two distinct elements. The lattice of this system is a distributive lattice with unique identities under the operations + and *. However, whenever $\underline{Z}$ has more than two distinct elements, there is not a unique complement such that $x*x = 0$ and $x+x = 1$, the definitions of the identities.

A Zadehian fuzzy logic and a Zadehian fuzzy algebra thus satisfy the following axioms:

1) idempotency $x+x=x$ and $x*x=x$

2) commutativity $x+y=y+x$ and $x*y=y*x$

3) associativity $(x+y)+z=x+(y+z)$ $(x*y)*z=x*(y*z)$

4) absorption $x+(x*y)=x$ $x*(x+y)=x$

5) distributivity $x+(y*z)=(x+y)*(x+z)$ $x*(y+z)=(x*y)+(x*z)$

Each of these axioms must satisfy the empirical evidence as supplied by the use of concepts or predicates in ordinary language or common thinking. With three of the five, this is readily demonstrated only insofar as counterexamples are exceedingly difficult to produce. The technique of using counterexamples can only prove which axioms are incompatible with the observation set. However, the profundity of examples and the difficulty of producing counterexamples may be taken as an indication of the validity of axioms as well.

LINGUISTIC EXAMPLES

The axiom of idempotency is generally valid in linguistic examples. The statement "John is tall" is taken to be equivalent to the statement "John is tall and John is tall" insofar as truth valuations are assigned. The absolute meaning implied by the statements may be different, however, in that repetition may provide emphasis for an inattentive listener in the conjunctive case or may attempt to communicate a lack of choice in the disjunctive case. Nonetheless, the truth valuation for all cases known to the author imply the validity of idempotency.

The axiom of associativity is similarly satisfied linguistically. In a string of conjuncts or disjuncts, variations in placement of parentheticals, underscoring, or other methods of grouping do not seem to alter the truth value of the particular predicate. Such punctuation is often used to convey the relative importance of a given conjunct or disjunct to arguments which either precede or follow the statement. For example, "John went to the store and Mary went to the store (and they took the dog)" may imply that "they took the dog" has little relevance to an argument which follows stating that John and Mary had the opportunity to rob the storekeeper. On the other hand, if the argument concerns the possibility that John had the dog attack the storekeeper, the underscoring of "John went to the store" and "they took the dog" might be more appropriate.

Consider the sentence "Tim is fat and Phil is fat or Tim is fat." Certainly the axiom of absorption is valid in this sentence. Although the analysis is a bit more complex, it is also valid for coupled predicates such as "She refused to see him whenever he came by and he came by or she refused to see him whenever he came by."

Coupled predicates can sometimes make a real difference in truth valuation of linguistic utterances. Difficulty is especially frequent when either the axioms of commutivity or distributivity are involved.

Consider the following example due to McCawley, 1975 [13]: "She got married and she got pregnant." This statement consists of a coupled predicate pair, which share the additional distinction of being order dependent. As such, truth valuation is not preserved under permutation: "She got pregnant and she got married."

An objection might be voiced over the particular example, claiming that "and" in this circumstance must be taken as meaning "and then." How then should the "and" in the following sentence be translated: "The apple and the orange are here and there, respectively." Applying the permutation once results in the sentence "The apple and the orange are there and here, respectively." Again, the truth valuation does not survive permutation about the conjunction. The agreement of elegance in modeling methodology [14], does not permit the use of heuristic rules of interpretation of operators. The axiomatic structure of the system must give explicit rules for the interpretation of the system observables and these rules must be as few in number as is consistent with the empirical system. A logic which contains ad hoc, heuristic rules of interpretation is not a logic, as there is no means for retionally determining the resolution of interpretive degeneracies. The key to each of the examples considered in this paragraph and the last is in the implicit ordering which underlies the coupling of the predicates. In the first example the ordering is temporal while in the second it is spatial. The ordering may be anything at all - the sensitivity to permutation remains.

In modeling any empirical system, care must be taken not to overtranslate. The standard practices of symbolic logic involve considerable "translating" of the observables (linguistic utterances) into the syntax of a Boolean logic. Unless it can be shown that a violation of the inherent structure of the observed system will not occur, such translation can reasonably be expected to obscure the very structure one wishes to expose.

In a similar empirical environment (quantum mechanics), Birkoff and von Neumann [1] have argued that order dependence such as that illustrated above is due to a failure of the distributive law. The lattice structure is that of a non-distributive, orthocomplemented, modular lattice. We suggest that a similar non-distributive lattice structure is more appropriate to ordinary language and common thought (which are indeed sensitive to ordering) than is a distributive lattice of fuzzy concepts.

Notice that there exist limited conditions under which a distributive lattice is an appropriate representation for natural linguistic propositions: there are a great many natural linguistic propositions which obey the distributive and the commutative laws. Formally, such local structures are known as the isles of Boole within the non-distributive lattice. The existence of these structures should not be construed as evidence that the global structure of cognitive processes is distributive. Quite the contrary. The structure of a non-distributive lattice is logically prior to that of a distributive lattice. Thus it is not possible to construct a non-distributive lattice from any combination of distributive lattices. This fact is referred to as an irreducibility postulate and was first noted by Birkoff in the classic work Lattice Theory [2].

NON-DISTRIBUTIVE TRUTH VALUATIONS

Dubois and Prade [3] have shown that, in any fuzzy set theory with [0,1] as a valuation set, either identity and distributivity, or excluded middle laws must be given up, these being mutually exclusive. We have shown that for quantum mechanics or natural linguistics (and probably cognitive processes in general), it is the distributive law which is not satisfied. It has also been shown [3], that for at least two fuzzy definitions of union and intersection which satisfy the conditions demanded

here of an empirical fuzzy logic, the negation defined as $N(p)=1-p$ is the only continuous operator satisfying the functional DeMorgan equations. A result follows which restricts empirical fuzzy logic even more.

Non-distributive truth valuations such as those which seem to occur in natural languages imply a certain restriction on the cardinality of the set Z in any fuzzy set theory or logic. It is easily demonstrated that a non-distributive logic cannot be mapped onto a truth valuation set having more than two elements where these elements correspond to numbers on the real number line.

THEOREM: There can be no coherent truth-valuation set Z of cardinality greater than two defined over the real numbers for a non-distributive logic.

Proof: Consider two propositions (predicates) $\underline{A}$ and $\underline{B}$ in a non-distributive logic $\underline{L}$ with corresponding truth valuations defined on the real numbers $\underline{R}$, $\underline{a}$ and $\underline{b}$. Coherence demands that a homomorphism exist between $\underline{R}$ and $\underline{L}$. Within $\underline{L}$ with operations $+$ and $*$ (conjunction and disjunction), neither $A+B=B+A$ nor $A*B=B*A$ are given. We choose $\underline{A},\underline{B}$ such that $A*B \neq B*A$. Define arbitrary truth functionals $f^*_L(A,B)$ and $f^*_R(a,b)$ in $\underline{L}$ and $\underline{R}$, respectively, such that $f^*_L(A,B)=f^*_R(a,b)$. In R, $f^*_R(a,b)=f^*_R(b,a)$. The existence of a homomorphism between $\underline{R}$ and $\underline{L}$ implies $f^*_R(b,a)= f^*_L(B,A)$ and thus that $f^*_L(A,B)=f^*_L(B,A)$. But this contradicts the assumption $A*B \neq B*A$ unless f^*_R is limited to two elements, namely the identities, e_+ and e_*. Hence it follows that the homomorphism does not exist unless the cardinality of Z is less than or equal to two.

CONCLUSIONS

Fuzzy logic requires considerable revision before it can be applied to any empirical science. We have raised several major difficulties which must be resolved. The questions of 1) a unique membership function,

2) the non-distributivity of the truth valuation set, 3) a definition of the implication which does not allow for the distributive law, and 4) a truth-valuation set which is not defined over the real numbers and allowing for coherence when the cardinality is greater than two. This fourth difficulty applies equally well to multi-valued logics and probability logics (having a finite probability measure) whenever they are applied to real situations which entail a non-distributive lattice.

The difficulties which Boolean logic experiences when applied to empirical systems might (intuitively) be resolved by the use of fuzzy variables, yet this same fuzziness leads to logical inconsistancies. And when one attempts to resolve these, one is led back to the usual Boolean logic without fuzziness except as an artifact [18].

We hope that these problems can be resolved. The promise of a truly empirical logic has been with us since Boole first introduced the propositional logic. Some variation of fuzzy logic may be the answer.

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